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WIND TUNNEL WALL INTERFERENCE.(U)

1978 E H DOWELL, D B BLISS

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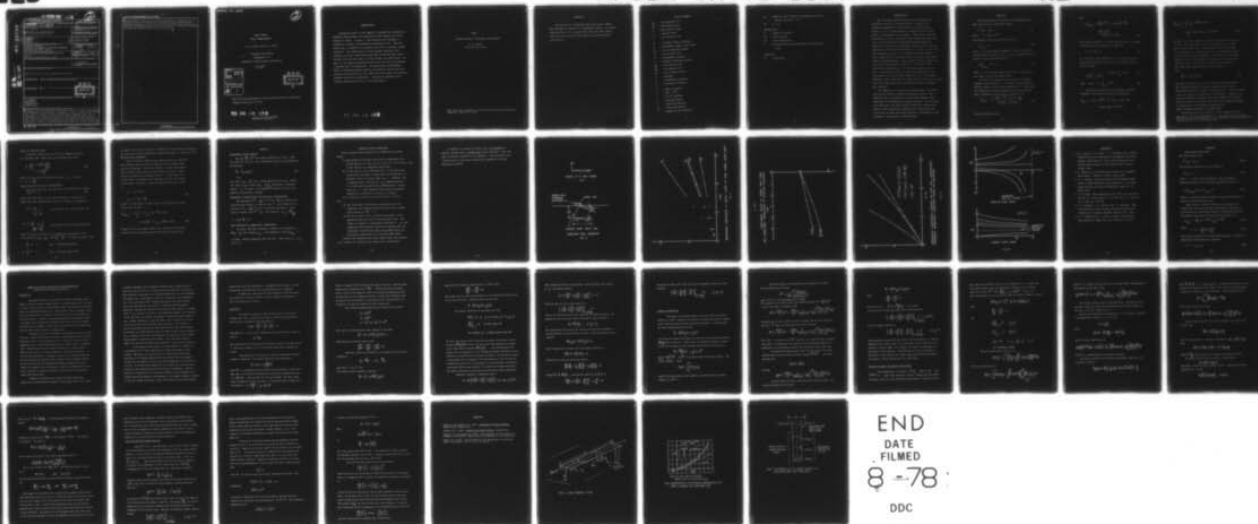
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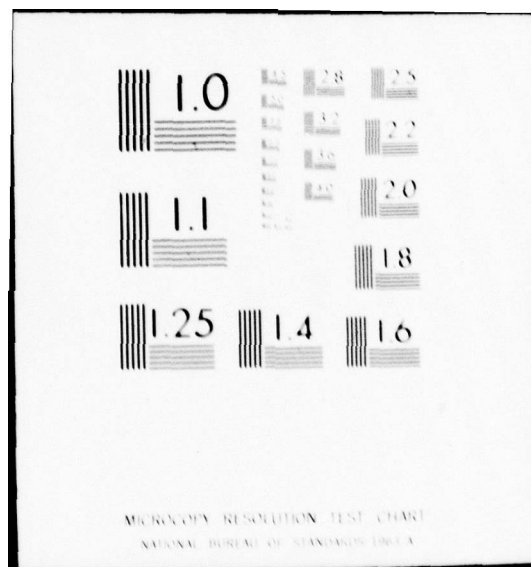
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An the direction of flow as well as flow through the slot perpendicular to the direction of flow. The theoretical relationship which is obtained between pressure differential across the slot and mass flow through the slot shows both the linear and quadratic regimes observed in experiments. Quantitative agreement with existing experimental data is also encouraging.

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WIND TUNNEL
WALL INTERFERENCE*

E. H. Dowell and D. B. Bliss

Set up
Princeton University
Department of
Mechanical and Aerospace Engineering

1978

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INTRODUCTION

A progress report on two aspects of theoretical research on improved understanding and reduction of wind tunnel wall interference is given. In part I, the compliant wall concept is studied for steady, two-dimensional supersonic flow. The compliant wall is modeled as a hinged plate with a torsional spring restraint. The combinations of flow dynamic pressure, wall geometry, and wall compliance which lead to minimum wall interference with the flow over a lifting airfoil are identified. In part II, an improved theoretical model for flow over and through a slotted wall is constructed. The theoretical relationship which is obtained between pressure differential across the slot and mass flow through the slot shows both the linear and quadratic regimes observed in experiments. Qualitative agreement with existing experimental data is very encouraging.

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PART I

A COMPLIANT WALL, SUPERSONIC WIND TUNNEL*

E. H. Dowell
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*This work was sponsored by the Air Force Office of Scientific
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ABSTRACT

The concept of a compliant wall wind tunnel, whose wall deforms in response to aerodynamic loading, is studied theoretically. It is argued that such walls have advantages for the design of wind tunnels with minimum wall interference.

LIST OF SYMBOLS

A_N	- see equation (17)
\tilde{A}_N	- see equation (25)
b	- wind tunnel height
c	- airfoil chord
f	- see equation (23)
K	- torsional spring stiffness
M	- aerodynamic moment about hinge
M_∞	- free stream Mach number
N	- see equation (17)
P	- aerodynamic pressure
s	- Laplace transform variable
T_N	- see equation (17)
\tilde{T}_N	- see equation (25)
U_∞	- free stream velocity
w	- downwash
x	- streamwise coordinate
z	- transverse coordinate
α	- angle of attack
β	$\equiv (M_\infty^2 - 1)^{1/2}$
λ	$\equiv \rho_\infty U_\infty^2 (\beta b)^2 / \beta K$
η	- wall deflection
ϕ	- velocity potential
ρ_∞	- free stream density
θ	- compliant wall rotation

θ_0 - compliant wall rotation corresponding to zero
spring twist or $\lambda = 0$

$\Delta\theta \equiv \theta - \theta_0$

Superscripts

* - Laplace transform

TE - trailing edge

ZI - zero interference

\sim - renormalization by multiplying or dividing by
 $c/\beta b$.

Subscript

D - divergence

INTRODUCTION

The self-correcting wind tunnel has attracted considerable interest in recent years as a concept for achieving minimum wall interference in a wind tunnel. Sears, et. al.¹, have presented an authoritative account which is recommended to the reader. Among their important conclusions is that a perforated wind tunnel wall (with control of the pressures in subdivided plenum chambers) is superior to control of impermeable, flexible walls such as suggested by Chevallier² and Goodyer³. In Ref. 2 and 3 and the discussion in Ref.1 it is assumed that the impermeable, flexible wall is deformed into an apriori determined shape of fixed magnitude. The selection of wall shape function and its magnitude is made so as to minimize wind tunnel wall interference. In the present paper this concept is broadened to allow at least the magnitude of the wall shape to vary with flow conditions, e.g. flow dynamic pressure. More generally the shape function might vary as well. As will be shown there are advantages to such a compliant wall concept.

For simplicity we consider two-dimensional, supersonic flow in the context of small perturbation theory. The compliant wall is modeled as rigid hinged segments whose compliance is determined by attached springs. As the reader will appreciate the basic concept allows for a much wider range of flow regimes and compliant wall characteristics at the cost of some increased complexity in mathematics and, ultimately, experimental implementation.

ANALYSIS

The fluid equation of motion for two-dimensional, small perturbation, potential flow is

$$\beta^2 \phi_{xx} - \phi_{zz} = 0 \quad (1)$$

where $\beta = (M_\infty^2 - 1)^{1/2}$

The boundary condition on the airfoil is

$$\phi_z \Big|_{z=0} = w \quad (2)$$

where w is the airfoil downwash. For an airfoil at steady angle of attack, α , $w = -U_\infty \alpha$. The boundary condition on the wind tunnel wall is

$$\phi_z \Big|_{z=b/2} = U_\infty \eta_x \quad (3)$$

where b is the wind tunnel height and η the deflection shape of the wind tunnel wall.

To solve (1), subject to (2) and (3), a Laplace Transform with respect to x is used.* Solving the resulting ordinary differential equation in z and using the transformed boundary conditions gives for the Laplace transform of the velocity potential, ϕ^* ,

$$\phi^* \Big|_{z=0} = -\frac{2}{\beta} \frac{\left[\frac{w^*}{s} e^{s \beta b/2} - U_\infty \eta^* \right] + \frac{w^*}{s \beta}}{e^{s \beta b/2} - e^{-s \beta b/2}} \quad (4)$$

*following Miles, Ref. 4.

and

$$\phi^* \Big|_{z=b/2} = \left[-\frac{w^*}{\beta s} e^{s\beta b/2} + U_\infty \eta^* \right] \coth \frac{s\beta b}{2} + \frac{w^*}{\beta s} \frac{e^{s\beta b/2}}{[e^{s\beta b/2} - e^{-s\beta b/2}]} \quad (5)$$

Our primary interest is in the fluid pressure, p , which is determined from Bernoulli's equation.

$$p = -\rho_\infty U_\infty \phi_x \quad (6)$$

To facilitate the inversions of (4) and (5) and subsequently determine p from (6), the following identities are used.

$$\frac{1}{e^{s\beta b/2} - e^{-s\beta b/2}} = e^{-s\beta b/2} \sum_{n=0}^{\infty} e^{-ns\beta b} \quad (7)$$

$$\coth \frac{s\beta b}{2} = 1 + 2 \sum_{n=1}^{\infty} e^{-ns\beta b} \quad (8)$$

Using these identities and standard inversion formulae, the following results are obtained.

$$p \Big|_{z=0} = -\rho_\infty U_\infty \left\{ -\frac{w(x)}{\beta} + \frac{2}{\beta} \sum_{n=1}^{\infty} [-w(x-n\beta b)] + U_\infty \eta_x \left(x - \frac{\beta b}{2} - (n-1)\beta b \right) \right\} \quad (9)$$

$$\begin{aligned}
p \Big|_{z=b/2} &= \frac{\rho_{\infty} U_{\infty}^2}{\beta} \sum_{n=1}^{\infty} w(x + \frac{\beta b}{2} - n\beta b) \\
&\quad - \frac{\rho_{\infty} U_{\infty}^2}{\beta} [\eta_x(x) + 2 \sum_{n=1}^{\infty} \eta_x(x - n\beta b)] \quad (10)
\end{aligned}$$

We shall use (10) subsequently; first we turn our attention to (9). The first term in (9) is the value of p if there were no wind tunnel wall (interference); the second term (first summation) is the effect of a rigid horizontal wall; the third term (second summation) is the effect of the wind tunnel wall deflection shape with respect to the horizontal reference line. For zero wall interference we choose η such that the second and third terms cancel.*

Thus

$$\eta_x^{ZI}(x) = \frac{w}{U_{\infty}} (x - \beta b/2) \quad (11)$$

One way of satisfying (11) is to have a rigid but moveable wall (perhaps composed of hinged segments) which can be controlled perfectly to obtain an appropriate η . This is essentially the concept of Chevallier² and Goodyer³. However, any practical moveable walls will have some compliance or flexibility; moreover as we shall show there may be some advantages to constructing a deliberately compliant wall.

SINGLE SEGMENT WITH HINGE AT LEADING EDGE:

For definiteness, we consider a zero thickness airfoil

* Note there is no wall interference if $c < \beta b$ since then the second and third terms are identically zero. Physically the waves reflected by the wall do not strike the airfoil.

at constant angle of attack, α , and a compliant wall consisting of a single rigid segment hinged at its leading edge. The compliance is due to an elastic torsional spring as shown in Fig. 2. It is assumed the leading edge of the wall segment begins at $x = \beta b/2$; in practice this might require a sliding leading edge to accomodate changing β or a multi-segment wall, each segment of which can be locked out or in as needed. The wall segment has its trailing edge at $x = c + \beta b/2$.

The wall angle for zero spring stretching is θ_0 ; the spring twist angle is $\Delta\theta$. Thus the total wall segment angle, θ , is given by

$$\theta = \theta_0 + \Delta\theta \quad (12)$$

For the geometry in question,

$$\begin{aligned} w &= -U_\infty \alpha & \text{for } 0 < x < c \\ \eta_x &= -\theta & \text{for } \frac{\beta b}{2} < x < \frac{\beta b}{2} + c \end{aligned} \quad (13)$$

The equation of equilibrium for wall segment is

$$M + K (\theta - \theta_0) = 0 \quad (14)$$

where K is the torsional spring constant and M the aerodynamic moment about the hinge point (positive in the direction of rotating the segment to close). M is given by

$$M = \int_{\beta b/2}^{c + \beta b/2} p \Big|_{z=b/2} (x - \beta b/2) dx \quad (15)$$

Substitution of (13) into (10) and the result into (15) gives

$$M = - K \lambda [\alpha A_N - \theta T_N] \quad (16)$$

where $\lambda \equiv \frac{\rho_{\infty} U_{\infty}^2 (\beta b)^2}{\beta K}$, is an aeroelastic coefficient,

$$A_N \equiv \sum_{n=1}^{N-1} n \frac{(2n-1)}{2}$$

$$+ N \left[\frac{c}{\beta b} - (N-1) \right]$$

$$\cdot \left[\frac{c}{\beta b} - (N-1) + (N-1) \right] ,$$

is an aerodynamic coefficient, and

$$T_N = \sum_{n=1}^{N-1} \frac{(2n-1)^2}{2}$$

$$+ (2N-1) \left[\frac{c}{\beta b} - (N-1) \right]$$

$$\cdot \left[\frac{c}{\beta b} - (N-1) + (N-1) \right] ,$$

is an aerodynamic coefficient, where the integer, N , is determined by

$$(N-1) < \frac{c}{\beta b} < N \quad (17)$$

Substitution of (16) into (14) and solving for θ gives

$$\frac{\theta}{\alpha} = \frac{[(\lambda T_N) \frac{A_N}{T_N} + \frac{\theta_0}{\alpha}]}{1 + \lambda T_N} \quad (18)$$

HINGE AT TRAILING EDGE:

A similar calculation for the wall segment hinged at its trailing edge rather than its leading edge gives

$$\frac{\theta}{\alpha} = \frac{\left[\frac{\theta_O}{\alpha} - (\lambda T_N^{TE}) \frac{A_N^{TE}}{T_N^{TE}} \right]}{1 - \lambda T_N^{TE}} \quad (19)$$

Note that aeroelastic divergence occurs, $\theta/\alpha \rightarrow \infty$, when $\lambda = \lambda_D \equiv 1/T_N^{TE}$.

CONDITIONS FOR ZERO WALL INTERFERENCE:

From (11) and (13), for zero wall interference we require $(\theta/\alpha)^{ZI} = 1$ (20)

Using (20) and (18) or (19), the aeroelastic coefficient at which zero wall interference occurs, λ^{ZI} , may be computed as

$$\lambda^{ZI} = \frac{\left[\frac{\theta_O}{\alpha} - 1 \right]}{T_N - A_N} \quad \text{for hinge at the leading edge (21)}$$

$$\text{or } \lambda^{ZI} = \frac{\left[1 - \theta_O/\alpha \right]}{T_N^{TE} - A_N^{TE}} \quad \text{for hinge at the trailing edge (22)}$$

Since λ^{ZI} must be positive to be physically meaningful (and as we shall see $T_N - A_N > 0$ and $T_N^{TE} - A_N^{TE} > 0$), then we require that

$$\frac{\theta_O}{\alpha} > 1 \quad \text{for a leading edge hinge}$$

$$\text{or } \frac{\theta_O}{\alpha} < 1 \quad \text{for a trailing edge hinge}$$

in order to be able to achieve a condition of zero wall interference by increasing the non-dimensional dynamic pressure λ , from 0 to λ^{ZI} .
 MULTIPLE WALL SEGMENTS:

Here we briefly indicate the generalization to multiple wall segments which would be required for an airfoil with arbitrary downwash. The basic concept is that each segment has a certain angle which is maintained relative to those of other segments by feedback control. In the limit of a large number of segments, one may think of the wall having a continuous variation. Hence the generalization of (13) is

$$w = - U_{\infty} \alpha f(x) \quad (23)$$

$$\eta_x = - \theta f(x - \frac{\beta b}{2})$$

where f is the shape of the airfoil (and wind tunnel wall).

Using (23) in (10),

$$p \Big|_{z=b/2} = - \frac{\rho_{\infty} U_{\infty}^2}{\beta} \left[\alpha \sum_{n=1}^{\infty} f(x + \frac{\beta b}{2} - n\beta b) - \theta f(x - \frac{\beta b}{2}) - 2\theta \sum_{n=1}^{\infty} f(x - \frac{\beta b}{2} - n\beta b) \right] \quad (24)$$

Using (24) in (15) again leads to an equation of the form given by (16) with generalized definitions of A_N and T_N .

RESULTS

AERODYNAMIC COEFFICIENTS:

$A_N, T_N, A_N^{TE}, T_N^{TE}$ are solely functions of $c/\beta b$. They are plotted in Fig.3. For convenience they are re-normalized by defining

$$\tilde{A}_N \equiv A_N / (c/\beta b)^2 \quad (25)$$

etc.

For large $c/\beta b$, \tilde{A}_N , etc., are proportional to $c/\beta b$. Hence for large $c/\beta b$ or small $\beta b/c$, \tilde{A}_N/\tilde{T}_N approaches a constant, namely .5. In Fig.4, the asymptotic behavior of the aerodynamic coefficients is shown.

AEROELASTIC COEFFICIENTS FOR ZERO WALL INTERFERENCE:

The variation $\tilde{\lambda}^{ZI} \equiv (\frac{c}{\beta b})^2 \lambda^{ZI}$ with $\frac{\beta b}{c}$ for leading and trailing edge hinges is shown in Fig. 5. Also shown for reference is $\tilde{\lambda}_D \equiv (\frac{c}{\beta b})^2 \lambda_D$ for a trailing edge hinge. Note that to insure that $\tilde{\lambda}^{ZI} < \tilde{\lambda}_D$, one requires $\theta_{o/\alpha} > \frac{\tilde{A}_N^{TE}}{\tilde{T}_N^{TE}}$ ($\rightarrow .5$ for $\frac{\beta b}{c} \rightarrow 0$).

WALL ROTATION VS AEROELASTIC COEFFICIENT:

In Fig.6, the wall rotation is shown as a function of $\tilde{\lambda}_{T_N} (\equiv \lambda_{T_N})$ for various $\theta_{o/\alpha}$. The asymptotic value $\tilde{A}_N/\tilde{T}_N = .5$ is used. Recall equations (18) and (19). Note that $\theta/\alpha = \theta_{o/\alpha}$ at $\tilde{\lambda} = 0$.

DISCUSSION AND CONCLUSIONS

First consider the rationale for a compliant wall wind tunnel.

- (1) Any movable wind tunnel wall will be inherently compliant anyway, so that one must take this into account even in the design of a nominally rigid wall.
- (2) If one were to try to maintain $\theta/\alpha = 1$ for zero wall interference by using a rigid wall, one would have to measure θ and α to insure that $\theta/\alpha = 1$. However, for a compliant wall, one may set θ_o/α appropriately at $\lambda = 0$ and then by controlling and measuring λ insure that $\theta/\alpha = 1$. This presumes the present aeroelastic analysis is sufficiently accurate, of course.

The principal conclusions from the present analysis are

- (3) The aerodynamic coefficients associated with the compliant wall loading rapidly approach their asymptotic values for $\frac{\beta b}{c} < .5$.
- (4) The wall segment with a leading edge hinge is more promising than one with a trailing edge hinge. The latter is subject to aeroelastic divergence. Also the compliant wall rotation varies more rapidly with the aeroelastic coefficient about its optimum value for zero wall interference for a trailing edge hinge.

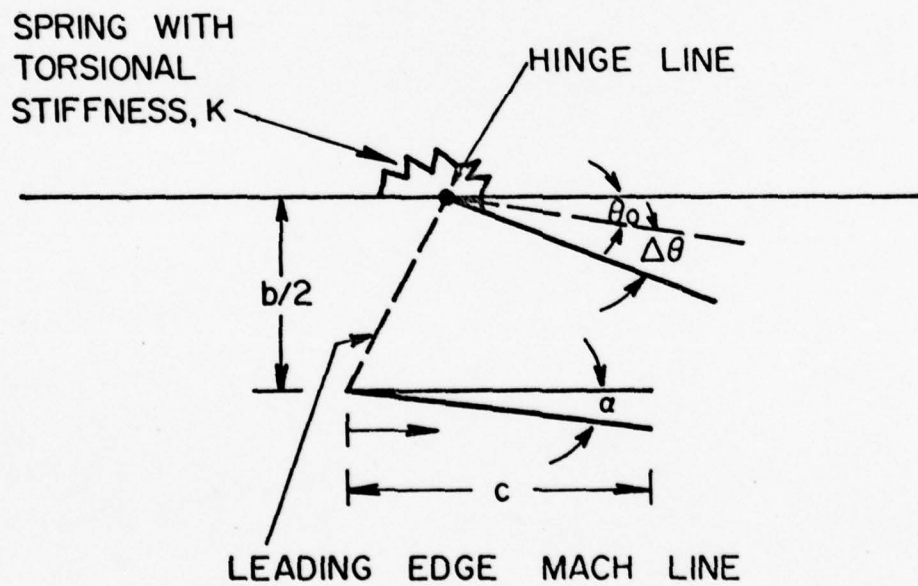
Overall the compliant wall wind tunnel shows real promise as a concept for reducing wind tunnel wall interference.

In closing, it should be noted, that, in principle, a similar concept using a porous wall may be employed. This concept is briefly explored in the Appendix. The difficulty with a porous wall is the basic uncertainty regarding its fluid mechanical behavior.



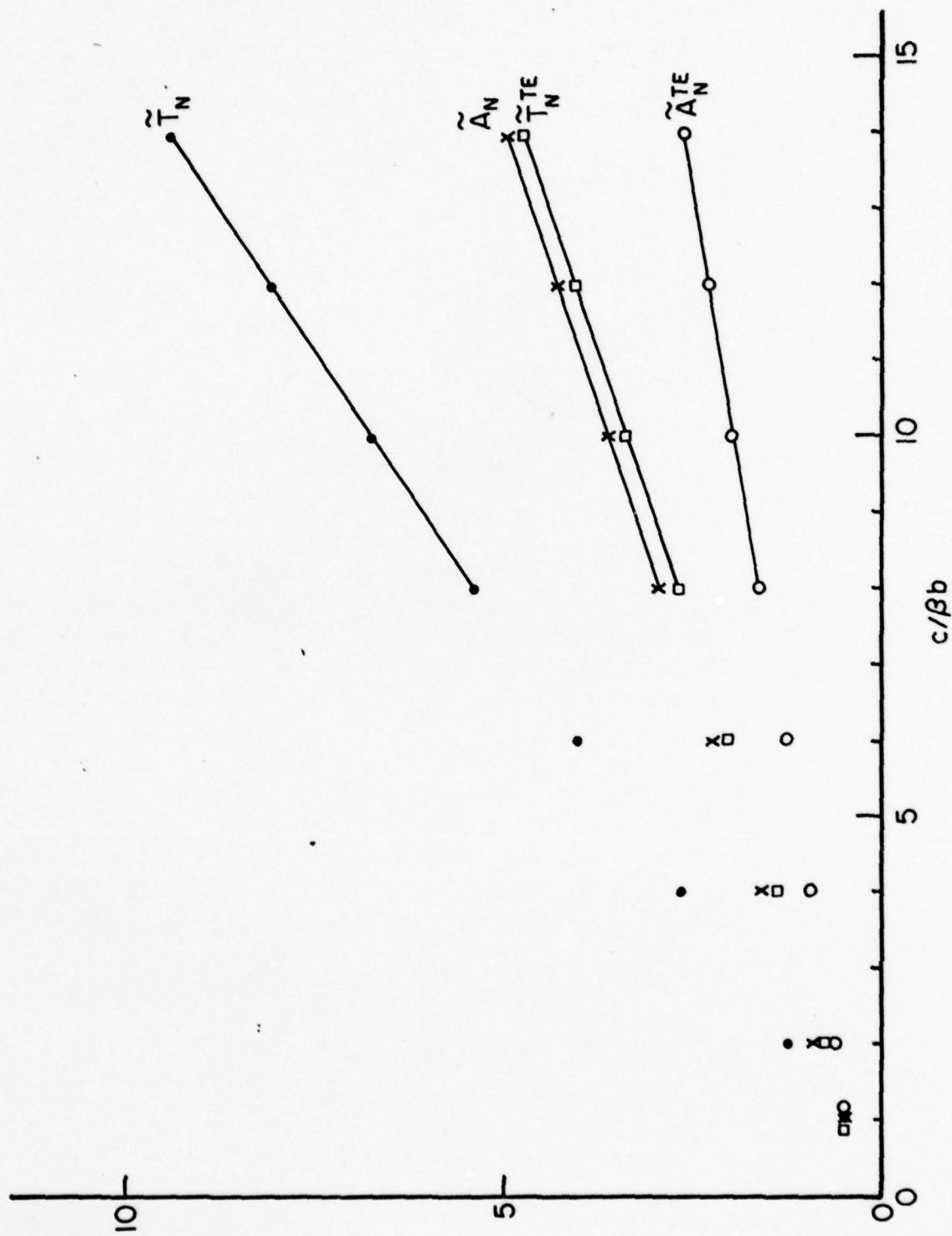
AIRFOIL IN A WIND TUNNEL

FIG. 1



COMPLIANT WALL GEOMETRY

FIG. 2



AERODYNAMIC COEFFICIENTS vs. AIRFOIL CHORD TO WIND TUNNEL HEIGHT RATIO

FIG. 3

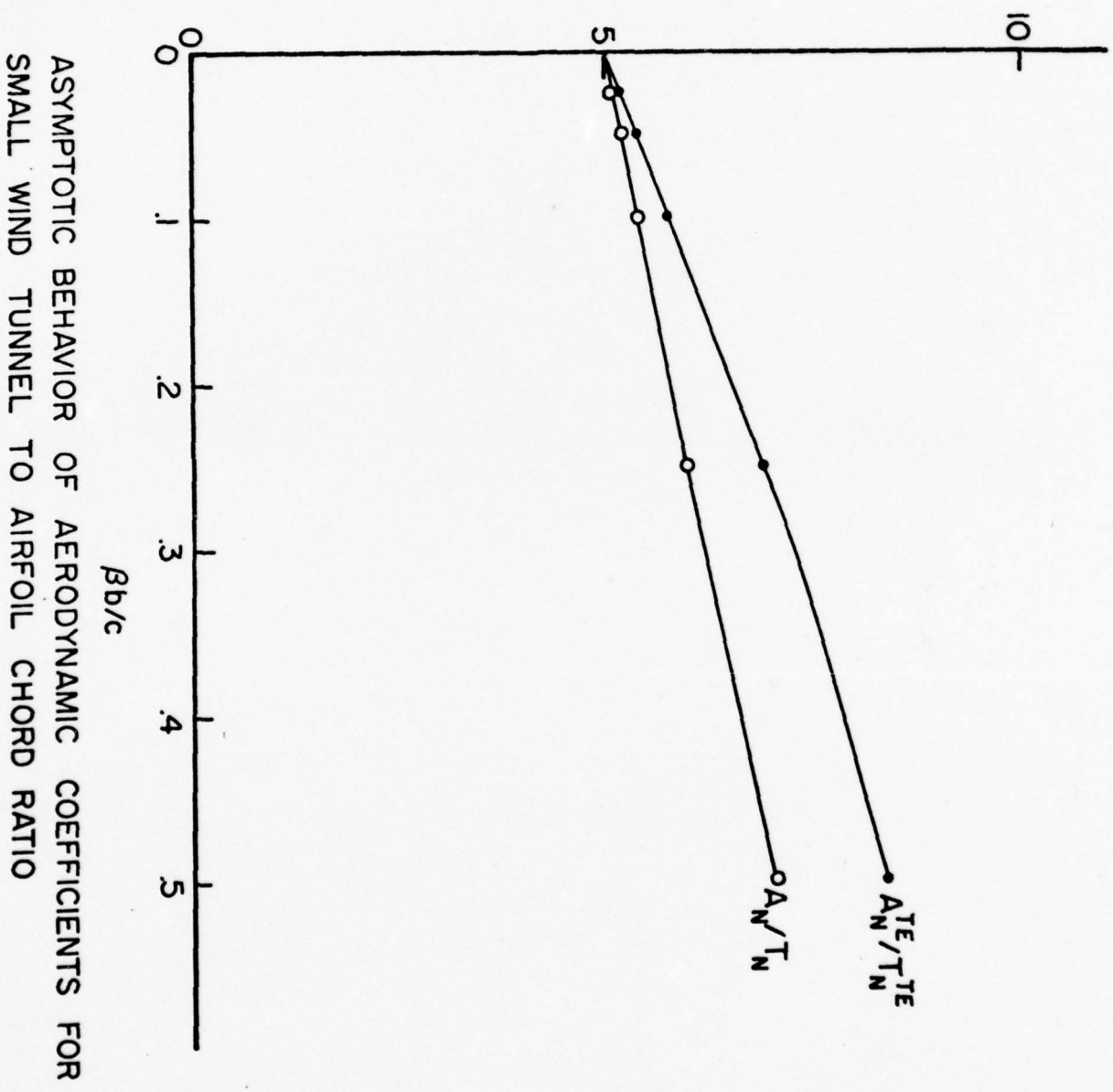
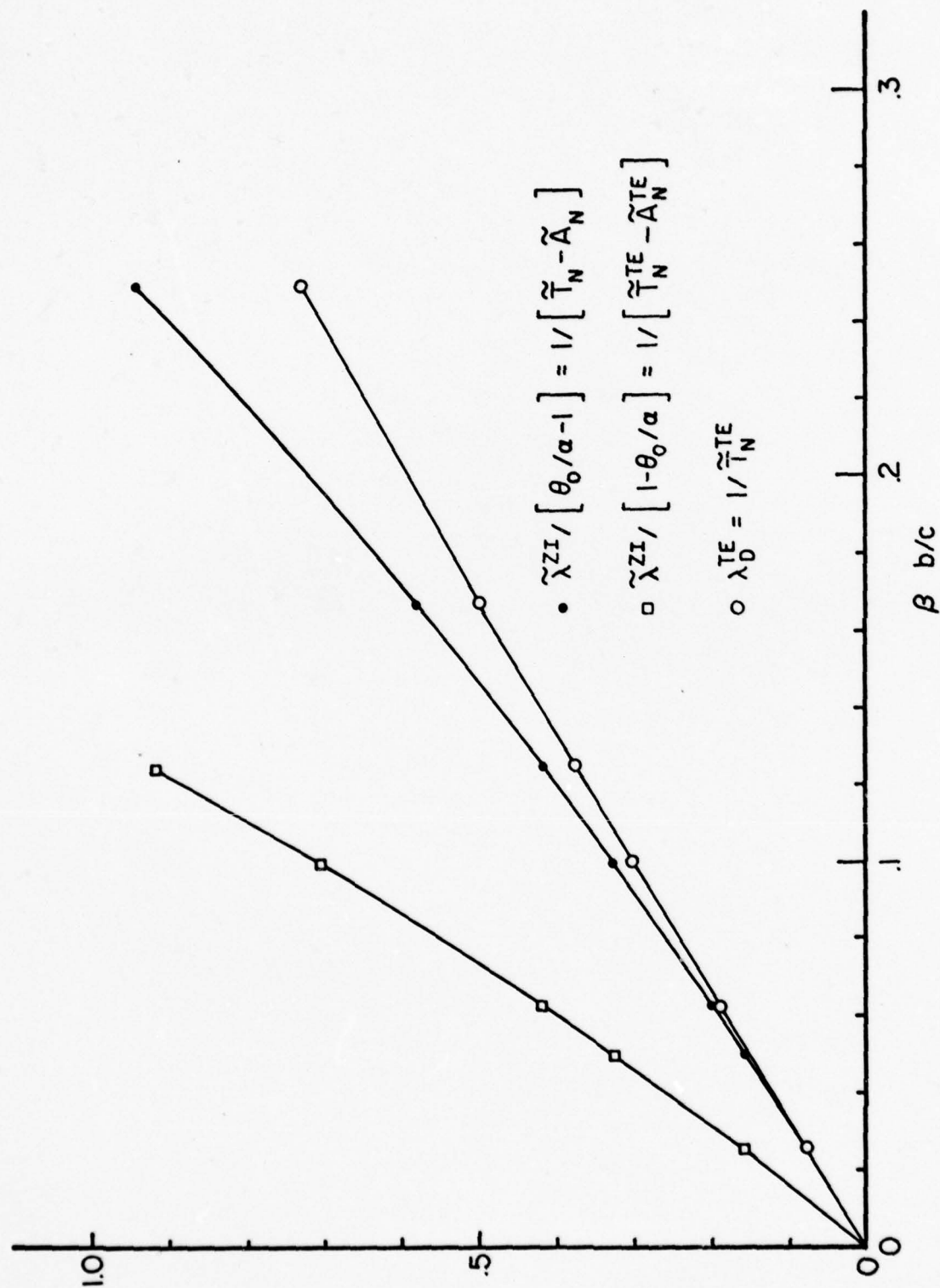


FIG. 4



AEROELASTIC COEFFICIENTS REQUIRED FOR ZERO WALL INTERFERENCE;
DIVERGENCE AEROELASTIC COEFFICIENT SHOWN FOR REFERENCE

FIG. 5

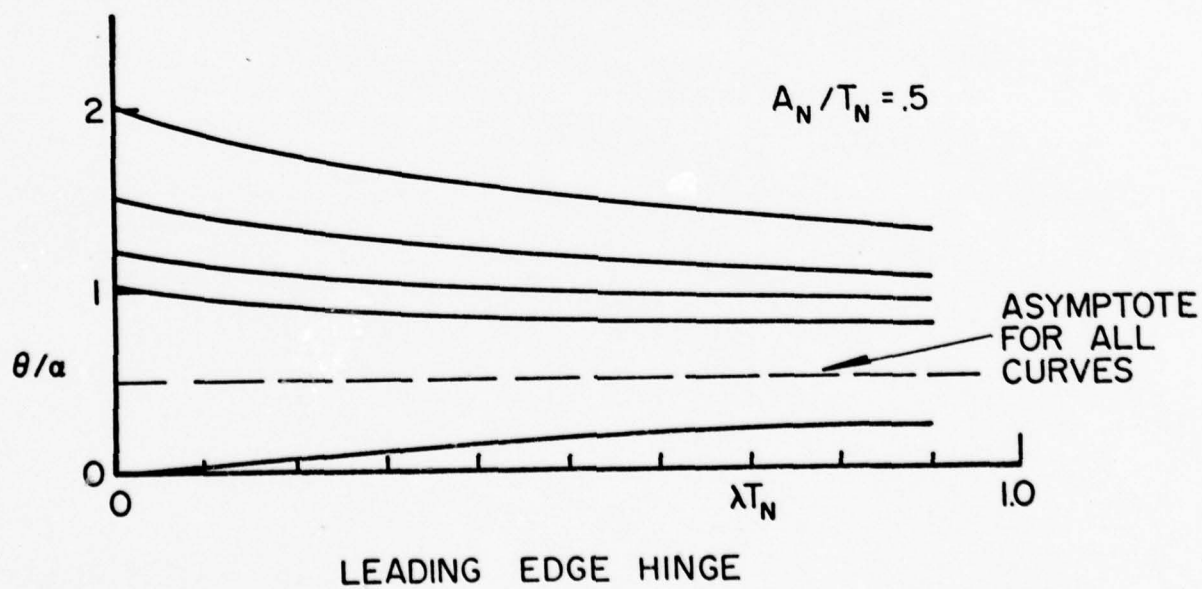
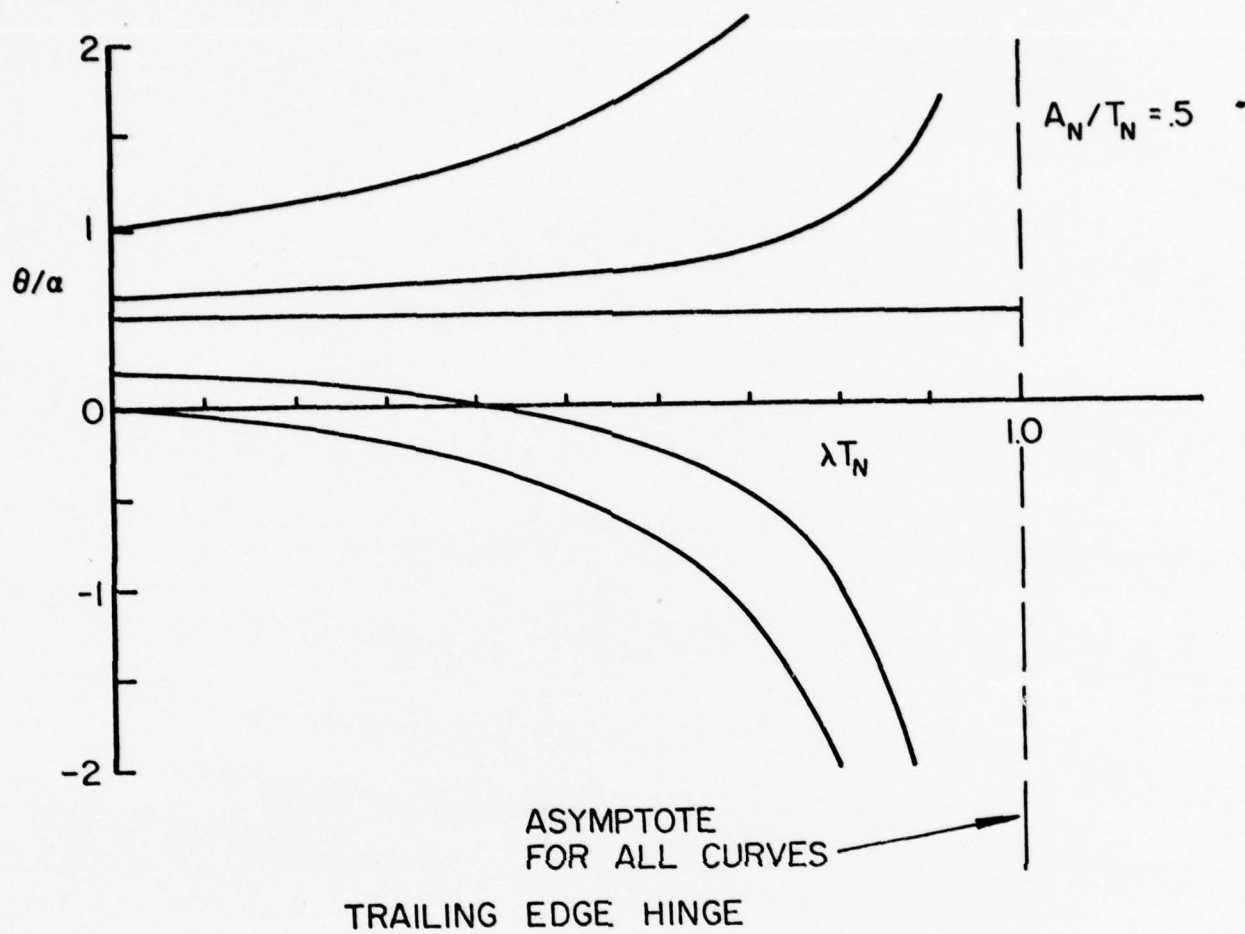


FIG. 6

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APPENDIX

Porous Wall Wind Tunnel

The field equation is

$$\beta^2 \phi_{xx} - \phi_{zz} = 0 \quad (A-1)$$

The boundary condition on the airfoil is

$$\phi_z \Big|_{z=0} = w \quad (A-2)$$

where w is the airfoil downwash. The traditional boundary condition on the porous wind tunnel wall is⁵

$$K_p \phi_z \Big|_{z=b/2} + \rho_\infty U_\infty \phi_x \Big|_{z=b/2} = 0 \quad (A-3)$$

where b is the wind tunnel height and K_p characterizes porosity of the wind tunnel wall.

Again using a Laplace Transform with respect to x , solving the resulting ordinary differential equation in z and using the transformed boundary conditions gives for the Laplace transform of the velocity potential, ϕ^* ,

$$\phi^* \Big|_{z=0} = - \frac{w^*}{\beta p} [1 + \gamma e^{-\beta p b}] \sum_{n=0}^{\infty} \gamma^n e^{-n \beta p b} \quad (A-4)$$

where

$$\gamma \equiv [\beta - \frac{\rho_\infty U_\infty}{K_p}] / [\beta + \frac{\rho_\infty U_\infty}{K_p}] \quad (A-5)$$

Our primary interest is in the fluid pressure, p , which is determined from Bernoulli's equation

$$p = - \rho_\infty U_\infty \phi_x \quad (A-6)$$

Using (A-4), (A-6) and standard inversion formulae, one obtains

$$p \Big|_{z=0} = \frac{\rho_{\infty} U_{\infty}}{\beta} \sum_{n=0}^{\infty} \gamma^n w(x - n\beta b) + \sum_{n=0}^{\infty} \gamma^{(n+1)} w(x - (n+1)\beta b) \quad (A-7)$$

By choosing $\gamma = 0$, (A-7) reduces to

$$p = \frac{\rho_{\infty} U_{\infty}}{\beta} w(x) \quad (A-8)$$

the well known Ackeret result corresponding to zero wall interference.

From (A-5), $\gamma = 0$ implies

$$\left(\frac{\rho_{\infty} U_{\infty}}{\beta} \right)^{ZI} = K_p \quad (A-9)$$

Thus by varying the wind tunnel density, for example, one could ensure that (A-9) is satisfied and zero wall interference is achieved. The crucial question is, to what degree does (A-3) accurately describe the fluid mechanics of a porous wall? It is this uncertainty regarding the adequacy of (A-3) which makes the compliant wind tunnel wall concept an attractive alternative.

The above result for a porous wall wind tunnel has been discussed by Maeder and Wood⁶ and was well known to earlier workers on wind tunnel wall interference.

PART II
A MODEL FOR THE FLOW THROUGH SLOTS IN WIND
TUNNEL WALLS*

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*This work was sponsored by the Air Force Office of Scientific
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A Model for the Flow through Slots in Wind Tunnel Walls

Donald B. Bliss, Princeton University

Introduction

Ventilated walls are used in the test sections of transonic wind tunnels in order to reduce blockage effects. In some designs, flow is allowed to leave or enter the test section through slender slots parallel to the flow which are distributed along the test section walls. The size, number, aspect ratio and other geometric properties of the slots may vary considerably for different tunnels. In order to include wind tunnel wall interference effects in transonic flow calculations the effect of the slots on the wall boundary condition must be included. A first step in this direction is to understand the flow field in the vicinity of a single slot.

The type of slot to be considered is illustrated in Figure 1, where the flow is shown leaving the test section. An experimental plot of pressure difference across a slot in uniform flow as a function of the mean velocity through the slot, suitably nondimensionalized, is shown in Fig. 2 (Goethert, 1967). When the mean velocity is small the behavior is linear, whereas when the mean velocity is large the behavior is more nearly quadratic. If the slot were subjected to a nonuniform flow over its length the behavior of the curve would be altered in a way related to the streamline curvature. The fact that the data for different Mach numbers collapses well onto a single curve suggests that the behavior of the slot is dominated by the subsonic cross-flow (y - z plane) and indicates that the slot can be analyzed using slender-body theory.

Numerous investigators have studied the flow through slots analytically and experimentally (Goethert, 1961; Berndt 1975, 1977). There

is general agreement that the quadratic behavior seen at higher values of mean slot velocity in Fig. 2 is associated with the head loss as the cross-flow separates at some point in its passage through the slot. The dependence of pressure drop on the square of velocity is the same as that encountered for a jet or an orifice flow. Perhaps surprisingly, the linear behavior that occurs when the mean slot velocity is small has not been properly explained. Attempts have been made to ascribe the linear region to a viscous effect in the vicinity of the slot. This explanation appears to be incorrect; in fact the effect of viscosity is probably to contribute another quadratic term. Another possibility is that the linear term is related to an interaction of the flow with the trailing edge of the slot. For instance, a blunt trailing edge would cause a flow deflection not unlike that produced by the holes in a perforated wall which are known to have a linear characteristic. However, if this were the mechanism, the slope of the linear region would probably be dependent on Mach number. Also, both sharp and blunt edged slots are known to exhibit linear behavior at low mean slot velocities.

In the analysis that follows, the linear behavior is shown to be a consequence of the correct application of slender body theory to this problem. Physically, the linear behavior is associated with the region downstream of the leading edge where the free surface flow is beginning to form but has not reached its final cross-flow configuration as an orifice or jet flow. If the mean flow through the slot is small, the cross-flow never reaches its fully developed state and the linear term dominates. When the mean flow is large most of the cross-flow resembles a fully developed jet or orifice flow and the quadratic term dominates. The somewhat analogous situation in classical slender body theory is that lift forces arise only from regions

in which the cross-flow experiences a streamwise rate of change, e.g. only from regions of the body where the cross-sectional area is changing.

In addition to showing how the linear behavior arises, the analysis also shows how to collapse the data in order to compare the features of slots of different aspect ratios.

Formulation

For reasons of simplicity only, the discussion is restricted to incompressible flow. Assume that the fluid is inviscid and irrotational, then the flow is governed by Laplace's equation

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

where $\Phi(x, y, z)$ is the velocity potential such that the velocity vector is given by

$$\vec{u} = \vec{\nabla} \Phi$$

The configuration of the slot and the coordinate system are as shown in Fig.

1. The slot is assumed to lie in an infinite plane below a uniform free stream.

The problem will be formulated in terms of the method of matched asymptotic expansions. The velocity potential can be expressed as

$$\Phi = U_{\infty} x + a \sqrt{\frac{\Delta p}{\frac{1}{2} \rho}} \varphi$$

Where φ is a nondimensional perturbation velocity potential. Notice that when there is no pressure difference across the plane containing the slot, i.e. $\Delta p = 0$, then the effect of the slot vanishes. The inner length scale is defined to be the slot width a . The relevant small parameter for the problem is $\epsilon = \sqrt{\frac{\Delta p}{\frac{1}{2} \rho}}$, $q = \frac{1}{2} \rho U_{\infty}^2$

which is a measure of the flow deflection angle in the slot. Then the outer length scale is defined to be $L = a/\epsilon$. Notice that the outer length scale is not the slot length, l , which is not really the appropriate scale for variations in the streamwise direction. In fact, this procedure could be used to solve the problem of a semi-infinite slot in which there is no possibility of using slot length as an outer length scale.

The following nondimensionalization is used for the variables:

$$\Phi = L U_{\infty} \bar{\Phi}^*$$

$$\vec{U} = U_{\infty} \vec{U}^*$$

$$X = L x^*, y = L y^*, z = L z^*$$

This leads to a non-dimensional outer expansion of the form:

$$\Phi^* = x^* + \epsilon^2 \varphi^*(x^*, y^*, z^*) + \dots$$

Substituting into Laplace's equation gives

$$\frac{\partial^2 \varphi^*}{\partial x^{*2}} + \frac{\partial^2 \varphi^*}{\partial y^{*2}} + \frac{\partial^2 \varphi^*}{\partial z^{*2}} = 0$$

The inner variables are defined by a stretching of the cross-flow coordinates

$$\bar{y} = y^*/\epsilon, \quad \bar{z} = z^*/\epsilon$$

(note that $\bar{y} = y/a$, $\bar{z} = z/a$)

Then assume an inner expansion of the form

$$\bar{\Phi} = x^* + \epsilon^2 \bar{\varphi}(x^*, \bar{y}, \bar{z})$$

Substituting into Laplace's equation gives, to lowest order

$$\frac{\partial^2 \bar{\varphi}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{\varphi}}{\partial \bar{z}^2} = 0$$

which shows that the inner region behaves as a two dimensional potential flow in the cross flow plane. Solutions must be of the form

$$\bar{\varphi} = \bar{\Phi}(\chi^*, \bar{y}, \bar{z}) + g(\chi^*)$$

The boundary conditions for the problem are that

$$\vec{\nabla} \varphi \rightarrow 0 \quad \text{as} \quad \rho_d \rightarrow \infty, \quad \text{where} \quad \rho_d^2 = \chi^2 + y^2 + z^2$$

$$\left. \frac{\partial \varphi}{\partial z} \right|_{z=0} = 0 \quad \text{on the rigid wall}$$

$$p = \text{constant} = p_s \quad \text{on the free surface in the slot}$$

The first and second of these conditions are applied to the outer velocity potential $\varphi^*(\chi^*, y^*, z^*)$, whereas the second and third conditions are applied to the inner velocity potential $\bar{\varphi}(\chi^*, \bar{y}, \bar{z})$. Any indeterminacy that remains after the application of the appropriate boundary conditions to each solution is resolved when the solutions are matched. For the outer solution, to the present order of expansion, the slot appears as a line of sinks distributed along the x-axis. The strength of this distribution is determined by matching with the inner solution, as is usual with slender body theory. The boundary conditions for the inner solution require closer examination.

Bernoulli's equation in dimensional form is

$$p + \frac{1}{2}\rho \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] = p_0 = p_\infty + \frac{1}{2}\rho U_\infty^2$$

After nondimensionalizing, re-expressing in inner variables, and setting

$p = p_s$ this equation becomes

$$\left(1 + \epsilon^2 \frac{\partial \bar{\Phi}}{\partial x^*}\right)^2 + \left(\epsilon \frac{\partial \bar{\Phi}}{\partial y}\right)^2 + \left(\epsilon \frac{\partial \bar{\Phi}}{\partial \bar{z}}\right)^2 - 1 = \epsilon^2$$

Retaining only the lowest order terms gives

$$\left[2 \frac{\partial \bar{\Phi}}{\partial x^*} + \left(\frac{\partial \bar{\Phi}}{\partial y}\right)^2 + \left(\frac{\partial \bar{\Phi}}{\partial \bar{z}}\right)^2 \right] \Big|_{\bar{z} = \bar{\eta}(x^*, y)} = 1$$

This is the condition that the free surface be at constant pressure. The unknown position of the free surface expressed in inner variables is

$$\bar{z} = \bar{\eta}(x^*, y), \quad -\frac{1}{2} \leq y \leq +\frac{1}{2}$$

The solution must also satisfy the condition that the flow be tangent to the free-surface at the free-surface location. If the free surface is described by

$$B(x, y, z) = \bar{B}(x^*, y, \bar{z}) = 0$$

then in dimensional coordinates the flow tangency condition is

$$\vec{\nabla} B \cdot \vec{u} = \vec{\nabla} B \cdot \vec{\nabla} \Phi = 0$$

Re-expressed in inner variables this becomes

$$\frac{\partial \bar{B}}{\partial x^*} \frac{\partial \bar{\Phi}}{\partial x^*} + \frac{1}{\epsilon^2} \frac{\partial \bar{B}}{\partial y} \frac{\partial \bar{\Phi}}{\partial y} + \frac{1}{\epsilon^2} \frac{\partial \bar{B}}{\partial \bar{z}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} = 0$$

Letting $\bar{B} = \bar{z} - \bar{\eta}(x^*, y)$, and using the expansion for $\bar{\Phi}$ gives

$$-\frac{\partial \bar{\eta}}{\partial x^*} \left(1 + \epsilon^2 \frac{\partial \bar{\Phi}}{\partial x^*}\right) - \frac{\partial \bar{\eta}}{\partial y} \frac{\partial \bar{\Phi}}{\partial y} + 1 \cdot \frac{\partial \bar{\Phi}}{\partial \bar{z}} = 0$$

Retaining the lowest order terms gives the flow tangency condition in inner variables

$$\left[\frac{\partial \bar{\eta}}{\partial x^*} + \frac{\partial \bar{\eta}}{\partial \bar{y}} \frac{\partial \bar{\varphi}}{\partial \bar{y}} - \frac{\partial \bar{\varphi}}{\partial \bar{z}} \right] \bigg|_{\bar{z} = \bar{\eta}(x^*, \bar{y})} = 0, \quad -\frac{1}{2} \leq \bar{y} \leq +\frac{1}{2}$$

Solutions and Matching

The general free surface problem in the slot would be extremely difficult to solve. Therefore this section is restricted to general comments on the solution structure and on the matching procedure. As stated before the inner solution must have the form

$$\bar{\varphi} = \bar{\phi}(x^*, \bar{y}, \bar{z}) + g(x^*)$$

As is usual for slender body theory, far from the slot the velocity potential $\bar{\phi}(x^*, \bar{y}, \bar{z})$ will look like a simple source or sink whose strength is related to the rate of change of cross-sectional area of the flow in the slot.

$$\bar{\varphi} \simeq -\frac{\bar{f}(x^*)}{\pi} \ln \bar{r} + g(x^*) \equiv \bar{\varphi}^o$$

Where $\bar{r} = \sqrt{\bar{y}^2 + \bar{z}^2}$ and $\bar{\varphi}^o$ is the outer limit of the inner solution. Thus $\bar{f}(x^*) = \bar{S}'(x^*)$, where

$$\bar{S}(x^*) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \bar{\eta}(x^*, \bar{y}) d\bar{y}$$

equals the cross-sectional area occupied by the flow in the slot (dimensionally $S = \alpha^2 \bar{S}$).

The outer solution can be expressed as an integral of a source-sink distribution over the slot

$$\varphi^* = + \frac{1}{2\pi} \int_0^{l^*} \frac{f^*(x_i^*) dx_i^*}{\sqrt{(x^* - x_i^*)^2 + r^{*2}}}$$

where $l^* = l/L$ is the nondimensional slot length.

This integral can be shown to have the following form when $r^* = \sqrt{y^{*2} + z^{*2}}$ is small (see Ashley and Landahl, 1965)

$$\varphi^* = -\frac{f^*(x^*)}{\pi} \ln r^* - \frac{f^*(x^*)}{2\pi} \ln \frac{1}{4x^*(l^* - x^*)} + \frac{1}{2\pi} \int_0^{l^*} \frac{f^*(x_i^*) - f^*(x^*)}{|x^* - x_i^*|} dx_i^*$$

Re-expressing the outer solution in inner variables and using the expression for small r^* gives the inner limit of the outer solution.

$$\varphi^{*i} = -\frac{f^*(x^*)}{\pi} \ln \bar{r} - \frac{f^*(x^*)}{2\pi} \ln \frac{\epsilon^2}{4x^*(l^* - x^*)} + \frac{1}{2\pi} \int_0^{l^*} \frac{f^*(x_i^*) - f^*(x^*)}{|x^* - x_i^*|} dx_i^*$$

Here $\ln \epsilon$ is treated as an $\mathcal{O}[1]$ quantity, as is sometimes done in slender-body theory. Alternatively, this problem can be treated more formally by a modification of the expansion procedure, but the results would be the same.

The matching procedure requires that the inner limit of the outer solution equal the outer limit of the inner solution: $\varphi^{*i} = \bar{\varphi}^*$. The matching shows that

$$\bar{S}'(x^*) = \bar{f}^*(x^*)$$

and then

$$g(x^*) = -\frac{\bar{S}'(x^*)}{2\pi} \ln \frac{\epsilon^2}{4x^*(l^* - x^*)} + \frac{1}{2\pi} \int_0^{l^*} \frac{\bar{S}'(x_i^*) - \bar{S}'(x^*)}{|x^* - x_i^*|} dx_i^*$$

Now the problem for flow in the inner region can be restated. The velocity potential is of the form

$$\bar{\varphi} = \bar{\phi}(x^*, \bar{y}, \bar{z}) + g(x^*)$$

where

$$\frac{\partial^2 \bar{\phi}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{\phi}}{\partial \bar{z}^2} = 0$$

Far from the slot $\bar{\phi} \rightarrow -\frac{S(x^*)}{\pi} \ln \bar{r} \quad \text{as } \bar{r} \rightarrow \infty.$

In the slot the constant pressure boundary condition becomes

$$\left[2 \frac{\partial \bar{\phi}}{\partial x^*} + \left(\frac{\partial \bar{\phi}}{\partial \bar{y}} \right)^2 + \left(\frac{\partial \bar{\phi}}{\partial \bar{z}} \right)^2 \right] \bigg|_{\bar{z} = \bar{\eta}(x^*, \bar{y})} = 1 - 2g'(x^*)$$

The flow tangency condition is

$$\left[\frac{\partial \bar{\eta}}{\partial x^*} + \frac{\partial \bar{\eta}}{\partial \bar{y}} \frac{\partial \bar{\phi}}{\partial \bar{y}} - \frac{\partial \bar{\phi}}{\partial \bar{z}} \right] \bigg|_{\bar{z} = \bar{\eta}(x^*, \bar{y})} = 0, \quad -\frac{1}{2} \leq \bar{y} \leq \frac{1}{2}$$

These equations constitute the final form of the inner problem. In addition, depending on the assumed shape of the free surface, other conditions may be required at the slot edges. For instance, the sides of a sharp edged slot may require a Kutta condition. The difficulty in solving the problem as posed above is that both the free surface shape and the velocity potential are unknown. Certain limiting cases are solved below.

Slot Flow for Small Free Surface Displacements

The nondimensional slot length is $l^* = \epsilon \frac{l}{\alpha}$. When $\epsilon \ll \frac{\alpha}{l}$ then $x^* \leq l^* \ll 1$. Under these conditions, since ϵ measures the flow deflection angle in the slot, the free surface displacement is sufficiently small

that the pressure boundary condition can be applied on $\bar{z} = 0$. Assume that the quadratic terms in the pressure boundary condition can be neglected (they are of order $(\lambda^*)^2$); thus $\frac{\partial \bar{\phi}}{\partial x^*} \Big|_{\bar{z}=0} = \frac{1}{2} - g'(\lambda^*)$, $-\frac{1}{2} \leq \bar{y} \leq +\frac{1}{2}$.

The following function will satisfy this simplified pressure boundary condition:

$$\bar{\phi}(\lambda^*, \bar{y}, \bar{z}) = \left(\frac{\lambda^*}{2} - g(\lambda^*) + C_1 \right) \phi_1(\bar{y}, \bar{z})$$

where

$$\frac{\partial^2 \phi_1}{\partial \bar{y}^2} + \frac{\partial^2 \phi_1}{\partial \bar{z}^2} = 0$$

and

$$\frac{\partial \phi_1}{\partial \bar{z}} \Big|_{\bar{z}=0} = 0, \quad |\bar{y}| > \frac{1}{2}$$

$$\phi_1 \Big|_{\bar{z}=0} = 1, \quad |\bar{y}| < \frac{1}{2}$$

$$\phi_1 \propto \ln \bar{r} \quad \text{as } \bar{r} = \sqrt{\bar{y}^2 + \bar{z}^2} \rightarrow \infty$$

$C_1 = \text{arbitrary constant}$

The free surface shape in the slot is

$$\bar{\eta}(\lambda^*, \bar{y}) = \int_0^{\lambda^*} \frac{\partial \bar{\phi}}{\partial \bar{z}} \Big|_{\bar{z}=0} d\lambda^* = \int_0^{\lambda^*} \left(\frac{\lambda^*}{2} - g(\lambda^*) + C_1 \right) \frac{\partial \phi_1(0, \bar{y})}{\partial \bar{z}} d\lambda^*$$

The cross-sectional area is

$$\bar{S}(\lambda^*) = \int_{-1/2}^{+1/2} \bar{\eta}(\lambda^*, \bar{y}) d\bar{y} = \int_0^{\lambda^*} \left(\frac{\lambda^*}{2} - g(\lambda^*) + C_1 \right) \underbrace{\left[\int_{-1/2}^{+1/2} \frac{\partial \phi_1(0, \bar{y})}{\partial \bar{z}} d\bar{y} \right]}_{\equiv \bar{w}} d\lambda^*$$

where \bar{w} is a constant of order unity (to be discussed later). Differentiating and substituting for $g(x^*)$ yields

$$\frac{1}{\bar{w}} \bar{S}'(x^*) = \frac{x^*}{2} + C_1 + \frac{\bar{S}'(x^*)}{2\pi} \ln \frac{\epsilon^2}{4x^*(l^* - x^*)} - \frac{1}{2\pi} \int_0^{l^*} \frac{\bar{S}'(x_i^*) - \bar{S}'(x_i)}{|x^* - x_i^*|} dx_i^*$$

This integral equation must be solved for $\bar{S}'(x^*)$. The presence of the arbitrary constant C_1 should allow the application of a Kutta condition at the front of the slot, namely $\bar{S}'(0) = 0$.

Now that the equation has been formulated, it is convenient to change to a new dimensionless variable $\hat{x} = x/l$, so that

$$x^* = \epsilon \frac{l}{a} \hat{x}$$

Then

$$\bar{S}'(x^*) = \bar{S}'(\hat{x}) \frac{d\hat{x}}{dx^*} = \bar{S}'(\hat{x}) / \epsilon \frac{l}{a}$$

and the integral equation becomes

$$\frac{1}{\pi} \ln \left(\frac{4l}{a} e^{\pi/\bar{w}} \right) \bar{S}'(\hat{x}) = \left(\epsilon \frac{l}{a} \right)^2 \left(\frac{\hat{x}}{2} + \hat{C}_1 \right) - \frac{\bar{S}'(\hat{x})}{2\pi} \ln [\hat{x}(1-\hat{x})] - \frac{1}{2\pi} \int_0^1 \frac{\bar{S}'(\hat{x}_i) - \bar{S}'(\hat{x})}{|\hat{x} - \hat{x}_i|} d\hat{x}_i$$

where \hat{C}_1 is a redefinition of the arbitrary constant.

The velocity potential which satisfies the boundary conditions on the crossflow is given by

$$\Phi_i(\bar{y}, \bar{z}) = \Re \left\{ + \frac{1}{\ln 4} \ln [-2\bar{z} + \sqrt{1 + (2\bar{z})^2}] - 1 \right\}$$

where $\bar{g} = \bar{z} + i\bar{y}$ is a complex quantity. This velocity potential is associated with the potential flow through a slit; the velocity field exhibits a square-root singularity at the side edges. A brief calculation gives

$$\bar{w} \equiv \int_{-1/2}^{+1/2} \frac{\partial \phi}{\partial \bar{z}}(0, \bar{y}) d\bar{y} = \pi / \ln A$$

The integral equation now becomes

$$\frac{1}{\pi} \left(\ln \frac{8l}{a} \right) \bar{S}'(\hat{x}) = \left(\epsilon \frac{l}{a} \right)^2 \left(\frac{\hat{x}}{2} + \hat{c}_1 \right) - \frac{\bar{S}'(\hat{x})}{2\pi} \ln[\hat{x}(1-\hat{x})] - \frac{1}{2\pi} \int_0^1 \frac{\bar{S}'(\hat{x}_1) - \bar{S}'(\hat{x})}{|\hat{x} - \hat{x}_1|} d\hat{x}_1$$

Clearly, from the form of this integral equation, the solution must have the form

$$\bar{S}(\hat{x}) = \left(\epsilon \frac{l}{a} \right)^2 \mathcal{L}(\hat{x}, \ln \frac{8l}{a})$$

Recall the dimensional relationships: $\hat{x} = 1$ for $x = l$ and $a^2 \bar{S}(\hat{x}) = S(x)$.

Thus

$$S(l) = \epsilon^2 l^2 \mathcal{L}(1, \ln \frac{8l}{a}) \equiv \epsilon^2 l^2 / G(\ln \frac{8l}{a})$$

where $\epsilon = \Delta p / q$ and the function G is defined by the above expression.

The volume flow rate from the slot is, to lowest order,

$$Q = U_{\infty} S = w_m a l$$

where w_m is the mean velocity through the slot. Substituting the above expression for S gives

$$U_{\infty} \frac{\Delta p}{q} l^2 / G(\ln \frac{8l}{a}) = w_m a l$$

or

$$\frac{\Delta p}{q} = \frac{a}{l} G\left(\ln \frac{8l}{a}\right) \frac{w_m}{U_\infty}$$

This shows a linear relationship between the pressure difference across the slot and the mean velocity through the slot. Recall that this relationship will hold only for $\epsilon \ll a/l$, i.e. only for small free surface displacements. The relationship between pressure difference and mean slot velocity can also be written as

$$\left(\frac{\Delta p}{q} \frac{l^2}{a^2}\right) = G\left(\ln \frac{8l}{a}\right) \left(\frac{w_m}{U_\infty} \frac{l}{a}\right)$$

Again, this linear relationship holds only for $\left(\frac{w_m}{U_\infty} \frac{l}{a}\right) \ll 1$, which corresponds physically to small free surface displacements compared to the slot width.

The slope of the pressure difference versus velocity curve is

$$\frac{a}{l} G\left(\ln \frac{8l}{a}\right)$$

The quantity a/l is the aspect ratio of the slot. Its appearance is analogous to the familiar result in slender wing theory that the lift curve slope is proportional to the aspect ratio. The fact that the argument of the function G is logarithmic suggests that its dependence on slot aspect ratio may be relatively weak. The integral equation must be solved to determine the function $G\left(\ln \frac{8l}{a}\right)$.

In order to estimate the type of results given by this analysis, an approximate solution to the integral equation is now obtained. For very small aspect ratio, a/l , the term involving $\bar{S}'(\chi)$ which has the coefficient $\ln \frac{8l}{a}$ will dominate. Then the equation is approximately

$$\frac{1}{\pi} \left(\ln \frac{8\ell}{a} \right) \bar{S}'(\hat{x}) \approx \left(\epsilon \frac{\ell}{a} \right)^2 \left(\frac{\hat{x}}{2} + \hat{C}_1 \right)$$

Physically, this equation balances the acceleration of fluid near the slot and the applied pressure difference Δp ; the effect of the source-sink distribution on the pressure distribution in the slot is neglected. (Actually this approximation is not formally consistent since it was previously assumed that $\epsilon \ll \gamma_\ell$ and $\ln \epsilon = O[1]$ in the derivation of the equation.) Setting $\hat{C}_1 = 0$, so that $\bar{S}'(0) = 0$, gives

$$\bar{S}'(\hat{x}) = \left(\epsilon \frac{\ell}{a} \right)^2 \pi \frac{\hat{x}}{\ln \frac{8\ell}{a}}$$

Integrating gives

$$\bar{S}(\hat{x}) = \left(\epsilon \frac{\ell}{a} \right)^2 \frac{\pi}{4} \frac{\hat{x}^2}{\ln \frac{8\ell}{a}}$$

from which

$$\bar{S}(1) = \left(\epsilon \frac{\ell}{a} \right)^2 \frac{\pi}{4} \frac{1}{\ln \frac{8\ell}{a}} \equiv \left(\epsilon \frac{\ell}{a} \right)^2 \frac{1}{G_1(\ln \frac{8\ell}{a})}$$

This result gives a function G_1 which can be viewed as a first approximation to the actual function G :

$$G_1\left(\ln \frac{8\ell}{a}\right) = \frac{4}{\pi} \ln \frac{8\ell}{a}$$

As a refinement, the neglected terms can be included by approximating $\bar{S}'(\hat{x})$ as a linear function, a form suggested by the approximate solution above. One possibility is simply to use this approximate solution in the terms previously neglected. Taking this approach and substituting into the integral equation gives

$$\frac{1}{\pi} \left(\ln \frac{8\ell}{a} \right) \bar{S}'(\hat{x}) = \left(\epsilon \frac{\ell}{a} \right)^2 \left(\frac{\hat{x}}{2} + \hat{C}_1 \right) - \left(\epsilon \frac{\ell}{a} \right)^2 \frac{\hat{x}}{4 \ln \frac{8\ell}{a}} \ln[\hat{x}(1-\hat{x})] - \left(\epsilon \frac{\ell}{a} \right)^2 \frac{(1-2\hat{x})}{4 \ln \frac{8\ell}{a}}$$

After choosing $\hat{C}_1 = \frac{1}{4} \ln \frac{8l}{a}$ to satisfy the Kutta condition, the equation becomes

$$\bar{S}'(\hat{x}) = \left(\epsilon \frac{l}{a}\right)^2 \frac{\pi}{2} \frac{\hat{x}}{\ln \frac{8l}{a}} \left[1 + \frac{1}{\ln \frac{8l}{a}} - \frac{1}{2 \ln \frac{8l}{a}} \ln[\hat{x}(1-\hat{x})] \right]$$

Although the expression for $\bar{S}'(\hat{x})$ is now singular at $\hat{x}=1$, the function is integrable. The result is

$$\bar{S}'(1) = \left(\epsilon \frac{l}{a}\right)^2 \frac{\pi}{4} \frac{1}{\ln \frac{8l}{a}} \left[1 + \frac{7}{4 \ln \frac{8l}{a}} \right]$$

Thus the function G_2 which is the second approximation for G is

$$G_2\left(\ln \frac{8l}{a}\right) = \frac{4}{\pi} \ln \frac{8l}{a} \left(\frac{4 \ln \frac{8l}{a}}{4 \ln \frac{8l}{a} + 7} \right)$$

For a slot of aspect ratio $a/l = 0.1$, the resulting values of G_1 and G_2 are

$$G_1 \approx 5.6 \quad \text{and} \quad G_2 \approx 4.0$$

The corresponding relationship between pressure difference and mass flow rate are

$$\left. \frac{\Delta p}{q} \right|_1 = 0.56 \frac{w_m}{U_\infty} \quad \text{and} \quad \left. \frac{\Delta p}{q} \right|_2 = 0.40 \frac{w_m}{U_\infty}$$

These numbers are considered to be in qualitative agreement with the available experimental data (e.g. see Fig. 2) where the slope of the pressure versus flow rate curve seems to fall in the range of less than 0.5 to greater than unity (Goethert, 1961). Aside from the approximate nature of the solution presented here, there are several other factors which could affect the accuracy of the calculation. The presence of a wall boundary layer and a shear layer in the slot could substantially lower the magnitude of the velocity in the slot.

Since transonic tunnel speeds are of primary interest, the effect of compressibility must also be considered. All of these effects are believed to act to increase the slope of the pressure drop versus flow rate curve. Regardless of these refinements, the analysis demonstrates that there is a (previously unrecognized) mechanism for linear behavior of this curve within the framework of an inviscid slender body theory analysis.

Fully Developed Flow Through the Slot

When $\frac{a}{l} \ll \epsilon \ll 1$ the cross-flow in the slot will closely resemble a fully developed jet or orifice flow (depending on the slot cross-sectional geometry). In this section, the problem formulation presented previously is shown to be consistent with this picture of the flow field. In this case $\chi^* = \epsilon \frac{l}{a} \gg 1$. Away from the ends of the slot the velocity potential is expected to be independent of χ^* to lowest order and $\bar{S}'(\chi^*) \approx \bar{S}'_c$, the constant local flow rate through the slot. Under these conditions,

$$g(\chi^*) \approx \frac{\bar{S}'_c}{4\pi} \ln \frac{\epsilon^2}{4\chi^*(1-\chi^*)}$$

Completely neglecting the integral that appears in the exact expression for $g(\chi^*)$ means that end effects are being ignored. Differentiating the above gives

$$g'(\chi^*) = \frac{\bar{S}'_c}{4\pi} \left[\frac{1}{\chi^*/l^*} - \frac{1}{1-\chi^*/l^*} \right] \frac{1}{l^*}$$

This quantity is $O[1/l^*]$ away from the ends of the slot and can be neglected in the constant pressure boundary condition. The quantity $\frac{\partial \bar{\Phi}}{\partial \chi^*}$ can also be neglected by the (consistent) assumption that the velocity potential is independent of χ^* to lowest order. Therefore, the pressure boundary condition becomes

$$\left[\left(\frac{\partial \bar{\Phi}}{\partial \bar{y}} \right)^2 + \left(\frac{\partial \bar{\Phi}}{\partial \bar{z}} \right)^2 \right] \bigg|_{\bar{z} = \bar{\eta}(\chi^*, \bar{y})} = 1 \quad -\frac{1}{2} < \bar{y} < +\frac{1}{2}$$

Thus, to good approximation, the velocity potential for the equivalent steady two-dimensional free surface flow can be used, provided the changes in free surface shape are confined to the region where the flow has become essentially parallel and the pressure is nearly constant ($= p_s$). This situation is illustrated schematically in Figure 3 for the case of a sharp edged slot.

To obtain the relationship between pressure difference and flow through the slot it is not necessary to consider the velocity potential in any detail. Suppose that the final width of the jet leaving a slot of width unity is σ . The precise value of σ depends on the slot geometry, but typically $\sigma = 0.611$ for a sharp edged slot, and $\sigma = 1.0$ for a thick slot without internal separation. Applying the boundary condition to the cross flow in the region downstream of the slot where the flow is almost parallel gives

$$\bar{w}_f^2 = 1$$

where \bar{w}_f is the final cross flow velocity downstream of the slot. Then

$$\bar{S}'(\chi^*) = \bar{S}'_c = \sigma \bar{w}_f = \sigma$$

Integrating,

$$\bar{S}(\chi^*) = \sigma \chi^*$$

A constant of integration that could be included in the above has been omitted since end effects are being neglected. Set $\chi^* = l^*$ and re-express in dimensional form:

$$S(\epsilon \frac{l}{a}) = \sigma \epsilon \frac{l}{a} a^2$$

As before, the flow rate through the slot is

$$Q = U_{\infty} S = w_m a l$$

Hence

$$U_{\infty} \sqrt{\frac{\Delta p}{\rho}} a l \sigma = w_m a l$$

or

$$\frac{\Delta p}{\rho} = \frac{1}{\sigma^2} \left(\frac{w_m}{U_{\infty}} \right)^2$$

For a sharp edge orifice $1/\sigma^2 \approx 2.68$. This equation was used to generate the "quadratic behavior" curve in Fig. 2 assuming a sharp edged orifice; note that the agreement with the data is quite good.

Note that the above result can also be rewritten as

$$\left(\frac{\Delta p}{\rho} \frac{l^2}{a^2} \right) = \frac{1}{\sigma^2} \left(\frac{w_m}{U_{\infty}} \frac{l}{a} \right)^2$$

Since the equation for linear behavior could also be written in an analogous form, it is suggested that all slender slot behavior can probably be expressed as

$$\left(\frac{\Delta p}{\rho} \frac{l^2}{a^2} \right) = F \left(\frac{w_m}{U_{\infty}} \frac{l}{a}, \ln \frac{l}{a} \right)$$

In fact this can be seen directly from the final statement of the inner problem. For the above form to hold, the cross-sectional shape of the slots must be geometrically similar and there must be no boundary layer effects. The parameter $\ln \frac{l}{a}$ may prove to have only a weak influence. It would be most interesting to plot an appropriate set of experimental data in the form

$$\left(\frac{\Delta p}{\rho} \frac{l^2}{a^2} \right) \text{ versus } \left(\frac{w_m}{U_{\infty}} \frac{l}{a} \right)$$

and see to what extent it collapses onto a single curve.

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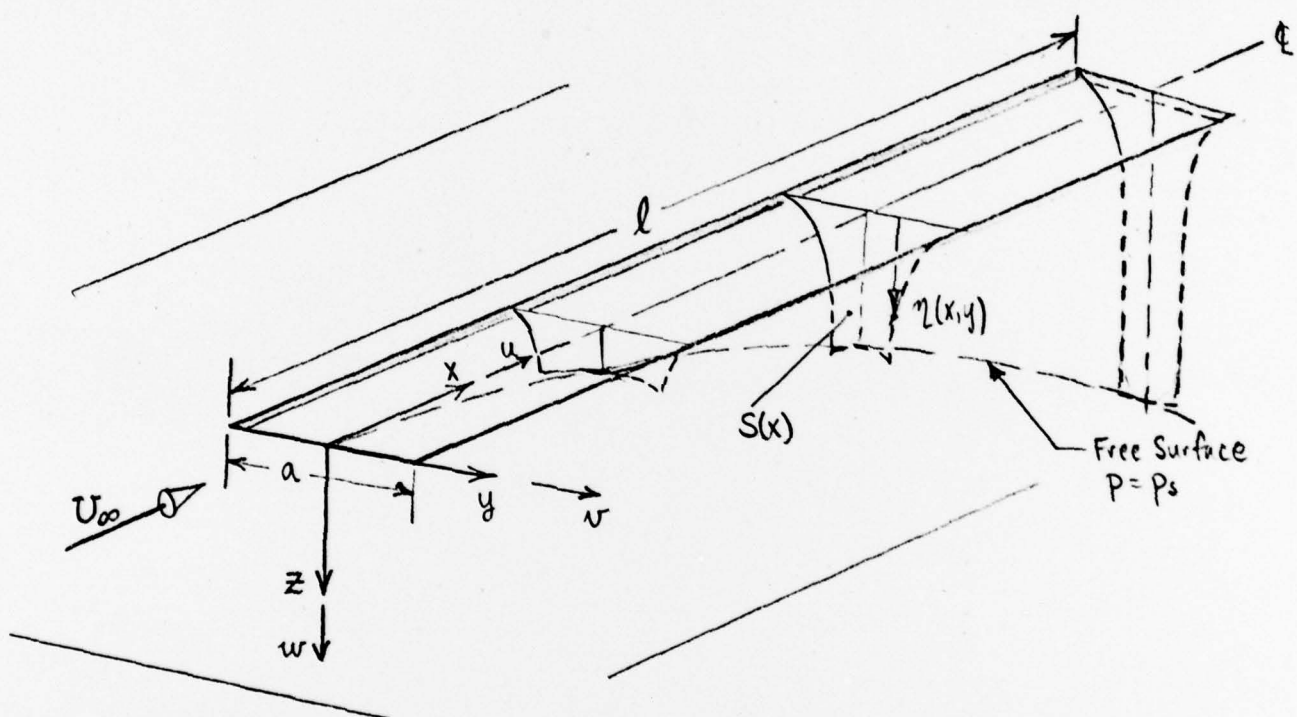
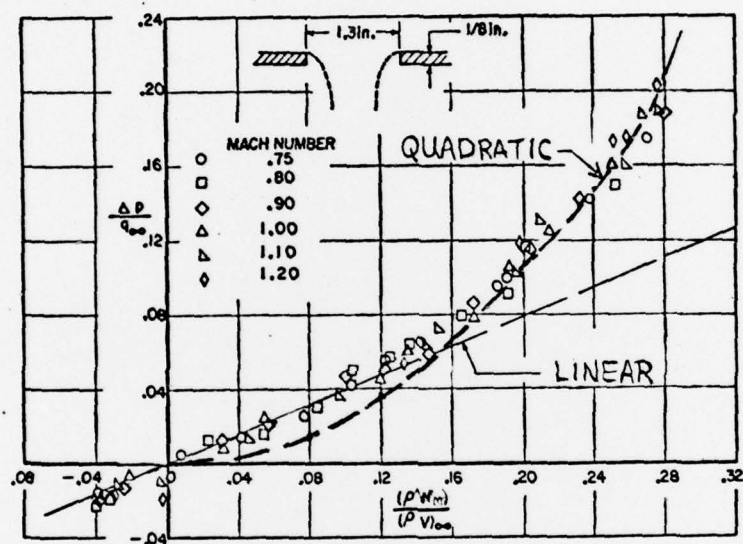


FIG.1 FLOW THROUGH A SLOT



Note: Linear curve chosen to best fit data.

Quadratic curve was computed from theory.

FIG.2 EXPERIMENTALLY DETERMINED CHARACTERISTICS OF A SINGLE SLENDER SLOT (GOETHERT, 1961)

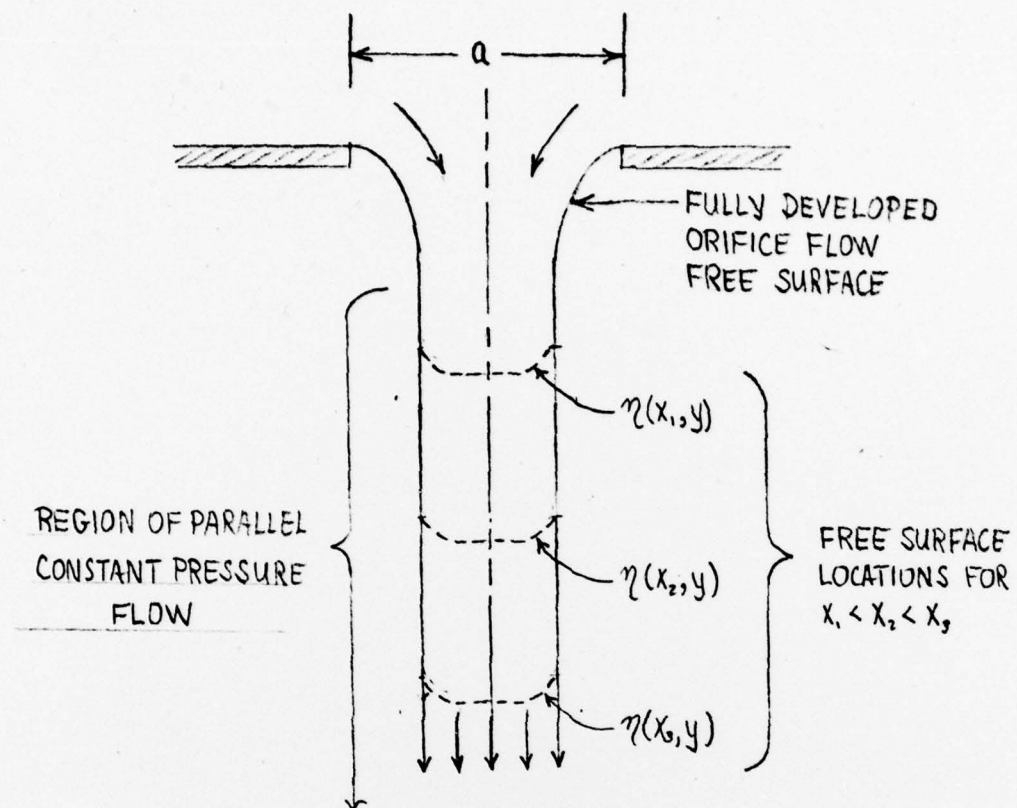


FIG.3 ILLUSTRATION OF FREE SURFACE BEHAVIOR IN A FULLY DEVELOPED SLOT CROSS-FLOW